Math 251
Final Review Key

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1. Evaluate the limit, if it exists.

   $$\lim_{x\to 1} \frac{x^3-1}{x^3-4x^2+3x}$$

Solution:

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-1}{x^3-4x^2+3x} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x(x-1)(x-3)} \\ &= \lim_{x\to 1} \frac{x^2+x+1}{x(x-3)} \\ &= \frac{(1)^2+(1)+1}{(1)(1-3)} = -\frac{3}{2}\end{aligned}$$

2. Evaluate the limit, if it exists.

   $$\lim_{s\to -\infty} \frac{\sin 3s}{\cos 2s}$$

Solution: Since sine and cosine each do not approach any value as s approaches $-\infty$, the limit does not exist.

3. Evaluate the limit, if it exists.

   $$\lim_{y\to \frac{4}{3}} \frac{3y-4}{|3y-4|}$$

Solution: Notice that

$$\lim_{y\to \frac{4}{3}^-} \frac{3y-4}{|3y-4|} = \lim_{y\to \frac{4}{3}^-} \frac{3y-4}{-(3y-4)} = \lim_{y\to \frac{4}{3}^-} -1 = -1$$

$$\lim_{y\to \frac{4}{3}^+} \frac{3y-4}{|3y-4|} = \lim_{y\to \frac{4}{3}^+} \frac{3y-4}{(3y-4)} = \lim_{y\to \frac{4}{3}^+} 1 = 1$$

Thus, the left- and right-handed limits are not equal. Therefore, the limit does not exist.

4. Find the equation of the line tangent to the curve $y = \dfrac{12x}{x+3} - 12x^3 + 2$ at the point $(1, -7)$.

Solution: We want $y - y_1 = m(x - x_1)$, so we need to find x_1, y_1, and m. Now, $x_1 = 1$ and $y_1 = -7$, so we need to find m, and $m = \frac{dy}{dx}\Big|_{x=1}$.

$$\frac{dy}{dx} = \frac{12(x+3) - 1(12x)}{(x+3)^2} - 36x^2$$
$$\frac{dy}{dx}\Big|_{x=1} = \frac{12(1+3) - 1(12(1))}{(1+3)^2} - 36(1)^2 = -\frac{135}{4}$$

Therefore, $y - y_1 = m(x - x_1)$ becomes $y + 7 = -\frac{135}{4}(x - 1)$.

5. Find the equation of the line tangent to the curve $y = \sin 2x + \cos 3x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution: We want $y - y_1 = m(x - x_1)$, so we need to find x_1, y_1, and m. Now, $x_1 = \frac{\pi}{2}$ and $y_1 = 0$, so we need to find m, and $m = \frac{dy}{dx}\Big|_{x=\frac{\pi}{2}}$.

$$\frac{dy}{dx} = 2\cos 2x - 3\sin 3x$$
$$\frac{dy}{dx}\Big|_{x=\frac{\pi}{2}} = 2\cos 2\frac{\pi}{2} - 3\sin 3\frac{\pi}{2} = 1$$

Therefore, $y - y_1 = m(x - x_1)$ becomes $y = x - \frac{\pi}{2}$.

6. Find the equation of the line tangent to the curve $x^2 + 4xy + y^2 - 13 = 0$ at the point $(2, 1)$.

Solution: We want $y - y_1 = m(x - x_1)$, so we need to find x_1, y_1, and m. Now, $x_1 = 2$ and $y_1 = 1$, so we need to find m, and $m = \frac{dy}{dx}\Big|_{(2,1)}$.

$$\frac{d}{dx}(x^2 + 4xy + y^2 - 13) = \frac{d}{dx}(0)$$
$$2x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(4x + 2y) = -2x - 4y$$
$$\frac{dy}{dx} = -\frac{2x + 4y}{4x + 2y}$$
$$\frac{dy}{dx}\Big|_{(\frac{\pi}{2},0)} = -\frac{2(2) + 4(1)}{4(2) + 2(1)} = -\frac{4}{5}$$

Therefore, $y - y_1 = m(x - x_1)$ becomes $y - 1 = -\frac{4}{5}(x - 2)$.

7. Differentiate $f(x) = \left(\frac{e^2 x}{2-x}\right)^3$.

Solution:

$$\begin{aligned} f'(x) &= 3\left(\frac{e^2x}{2-x}\right)^2 \frac{d}{dx}\left(\frac{e^2x}{2-x}\right) \\ &= 3\left(\frac{e^2x}{2-x}\right)^2 \left(\frac{e^2(2-x)-(-1)(e^2x)}{(2-x)^2}\right) \\ &= \frac{3e^4x^2}{(2-x)^2}\left(\frac{2e^2-e^2x+e^2x}{(2-x)^2}\right) = \frac{6e^6x^2}{(2-x)^4} \end{aligned}$$

8. Find the derivative of $y = (x-3)^2 \sin(2x)$.

Solution:

$$\frac{dy}{dx} = 2(x-3)\sin 2x + 2\cos(2x)(x-3)^2 = 2(x-3)(\sin 2x + (x-3)\cos 2x)$$

9. Find the derivative of $y = (4x^2 - 3x + 2)(\tan^2 x)$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= (8x-3)\tan^2 x + 2\tan x(\sec^2 x)(4x^2-3x+2) \\ &= \tan x\left[(8x-3)\tan x + 2\sec^2 x(4x^2-3x+2)\right] \end{aligned}$$

10. Find the derivative of $F(x) = \frac{\arctan x}{\sqrt{1-x^2}}$

Solution:

$$\begin{aligned} F'(x) &= \frac{\frac{1}{1+x^2}\sqrt{1-x^2} - \arctan x\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\right)}{(\sqrt{1-x^2})^2} \\ &= \frac{\frac{(1-x^2)^{\frac{1}{2}}}{1+x^2} + \frac{x\arctan x}{(1-x^2)^{\frac{1}{2}}}}{1-x^2} \\ &= \frac{(1-x^2)^{\frac{3}{2}}}{(1+x^2)(1-x^2)} + \frac{x\arctan x}{(1-x^2)^{\frac{3}{2}}} \\ &= \frac{(1-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}}{(1+x^2)(1-x^2)^{\frac{3}{2}}} + \frac{(1+x^2)x\arctan x}{(1+x^2)(1-x^2)^{\frac{3}{2}}} \\ &= \frac{1-x^2+(1+x^2)x\arctan x}{(1+x^2)(1-x^2)^{\frac{3}{2}}} \end{aligned}$$

11. Find $f''(x)$ if $f(x) = \sin x - 2x^2 + \ln x$.

Solution:

$$f'(x) = \cos x - 4x + \frac{1}{x}$$
$$f''(x) = -\sin x - 4 - \frac{1}{x^2}$$

12. Find the derivative of $f(t) = (\sin t - \cos t)^3 10^t$

Solution:

$$\begin{aligned} f'(t) &= 3(\sin t - \cos t)^2(\cos t + \sin t)10^t + 10^t(\ln 10)(\sin t - \cos t)^3 \\ &= 10^t(\sin t - \cos t)^2(3\cos t + 3\sin t + \ln 10 \sin t - \ln 10 \cos t) \end{aligned}$$

13. A particle moves along a straight line. The position of a particle is given by $s(t) = 3t^2 - 22t + 24$, where s is measured in meters and t is measured in seconds. Find

(a) The velocity at time t $v(t) =$____________

(b) The acceleration at time t $a(t) =$____________

(c) The velocity of the particle at 3 seconds ____________

(d) The time(s) when the particle is not moving ____________

(e) The position of the object when the acceleration is 0 ____________

Solution:

(a) $v(t) = s'(t) = 6t - 22$

(b) $a(t) = v'(t) = 6$

(c) $v(3) = -4$ m/s

(d) $v(t) = 0 \Longleftrightarrow 6t - 22 = 0 \Longleftrightarrow t = \frac{22}{6}$ seconds $= \frac{11}{3}$ seconds

(e) $a(t) = 6,$ so acceleration is never 0.

14. Use logarithmic differentiation to find $\frac{dy}{dx}$ if $y = x^{\arctan x}$, then differentiate.

Solution: Apply ln to both sides to get

$$\begin{aligned} y &= x^{\arctan x} \\ \ln y &= \ln x^{\arctan x} \\ \ln y &= \arctan x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}(\arctan x)\ln x + \frac{d}{dx}(\ln x)\arctan x \\ \frac{1}{y}\frac{dy}{dx} &= \frac{1}{1+x^2} \cdot \ln x + \frac{1}{x} \cdot \arctan x \\ \frac{dy}{dx} &= \left(\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right) y \\ \frac{dy}{dx} &= \left(\frac{\ln x}{1+x^2} + \frac{\arctan x}{x}\right) x^{\arctan x} \end{aligned}$$