12.9 The Jacobian

It is often useful to change variables in an expression to better understand how a quantity works. We have seen this in several circumstances.

$$\int_a^b f(x)\ dx = \int_{g(a)}^{g(b)} f(g(u))\ g'(u)\ du$$

$$\iint_R f(x,y)\ dA = \iint_S f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$$

$$\iiint_E f(x,y,z)\ dV = \iiint_W f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \rho^2\sin\varphi\ d\rho\ d\theta\ d\varphi$$

12.9.1 Transformations

Definition

A **transformation** T from a set D to another set D' is a rule that assigns to each element of D an element of D'.

- The set D is called the **domain** of T.
- The set D' is called the **codomain of** T.
- If $u \in D$, then the corresponding element of the codomain $T(u)$ is called the **image of** u.
- If E is a subset of D, then the set of all images of the elements of E is called the **image of** E.
- If no two elements of D have the same image, then T is called a **one-to-one transformation**.
- If T is a one-to-one transformation from D to D' such that $T(\alpha) = \beta$, then the transformation S from D' to D such that $S(\beta) = \alpha$ is called the **inverse transformation of** T, and we notate it $S = T^{-1}$.

Definition

A transformation T from the uv-plane to the xy-plane by the rule $T(u,v) = (x,y)$, where

$$x = g(u,v) \qquad , \qquad y = h(u,v)$$

is called a C^1 **transformation** if g, h have continuous first-order partial derivatives.

Example 1. Let T be a transformation from the uv-plane to the xy-plane defined by

$$g(u,v) = u^3 - v^3 \qquad , \qquad h(u,v) = -3u^2v + 3uv^2$$

Determine if T is a C^1 transformation, then find the image of the square $S = \{(u,v) \mid u \in [0,1], v = [0,1]\}$.

12.9.2 The Jacobian

Definition

Let T be a transformation from the uv-plane to the xy-plane determined by $x = g(u,v)$ and $y = h(u,v)$. The **Jacobian** of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem

Suppose that T is a C^1 transformation from a region S in the uv-plane onto a region R in the xy-plane such that the Jacobian is nonzero. Further suppose that

- f is continuous on R,
- R and S are type I or II plane regions,
- T is one-to-one except possibly on the boundary of S.

Then

$$\iint\limits_R f(x,y)\ dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

Example 2. Let R be the trapezoidal region with vertices $(1,0),(2,0),(0,-2),(0,-1)$. Evaluate $\iint\limits_R e^{\frac{x+y}{x-y}}\ dA$.

Definition

Let T be a transformation from the uwv-space to the xyz-space determined by $x = g(u,v,w)$, $y = h(u,v,w)$, and $z = k(u,v,w)$. The **Jacobian** of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem

Suppose that T is a C^1 transformation from a region S in the uvw-space onto a region R in the xyz-space such that the Jacobian is nonzero. Further suppose that

- f is continuous on R,
- R and S are type I, II, or III space regions,
- T is one-to-one except possibly on the boundary of S.

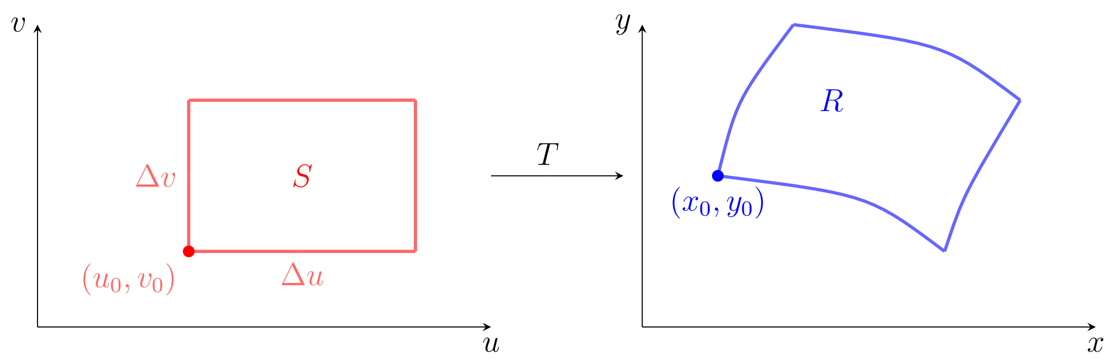
Then

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Example 3. Use the transformation $x = u^2$, $y = v^2$, and $z = w^2$ to find the volume of the region bounded by the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

12.9.3 Rationale for Using the Jacobian to Change Variables in a Double Integral

C^1 transformations are useful for transforming the region of integration in a double integral. Suppose S is a small rectangle in the uv-plane of size $\Delta u \times \Delta v$, and the lower-left corner of S is the point (u_0, v_0). Let T be a transformation from the uv-plane to the xy-plane, and let R be the image of S.



Now, if (u_0, v_0) is a corner of R, then the image of this corner is $(x_0, y_0) = T(u_0, v_0)$. Moreover, if $(u, v) \in R$, then

$$\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$$

is the position vector for the image of (u, v).

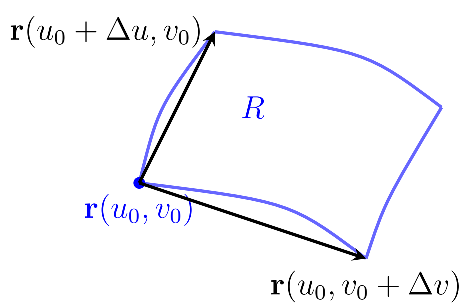
We can use this vector equation to study the boundaries of R. First, the left edge of S is (u_0, v). The corresponding boundary of R is $\mathbf{r}(u_0, v)$. Moreover, the tangent vector to this boundary at (x_0, y_0) is

$$\mathbf{r}_u = g_u(u_0, v_0)\mathbf{i} + h_u(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}$$

Similarly, the bottom edge of S is (u, v_0), the corresponding boundary of R is $\mathbf{r}(u, v_0)$, and the tangent vector to this boundary at (x_0, y_0) is

$$\mathbf{r}_v = g_v(u_0, v_0)\mathbf{i} + h_v(u_0, v_0)\mathbf{j} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

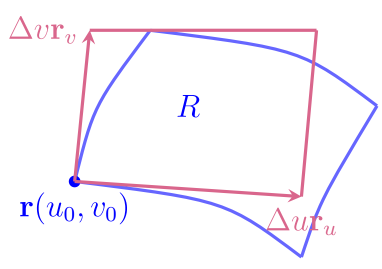
Similarly to how we did with exploring surface area, we can approximate the image region $R = T(S)$ by a parallelogram determined by $\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)$ and $\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)$.



Now, note that

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

So $\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \approx \Delta u \mathbf{r}_u$. Similarly $\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \approx \Delta v \mathbf{r}_v$.



It follows that the area of R is approximated by the area of the parallelogram determined by $\Delta u\mathbf{r}_u$ and $\Delta v\mathbf{r}_v$.

A property of cross products is that $|a\mathbf{u} \times b\mathbf{v}| = |\mathbf{u} \times \mathbf{v}||a||b|$. In particular,

$$|(\Delta u\mathbf{r}_u) \times (\Delta v\mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\Delta u\Delta v$$

Computing this cross product, we obtain

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} \\
&= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}
\end{aligned}$$

It follows that the approximate area of R is $\Delta A \approx |\frac{\partial(x,y)}{\partial(u,v)}|\Delta u\Delta v$, where the Jacobian is evaluated at (u_0, v_0).

Now, if we divide any region S in the uv-plane into rectangles S_{ij} and call their images in the xy-plane R_{ij}, then

$$\begin{aligned}
\iint_R f(x,y)\ dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\ \Delta A \\
&\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v
\end{aligned}$$

where we evaluate the Jacobian at (u_i, v_j). As $m, n \to \infty$,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta v \to \iint_S f(g(u,v), h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$